Int. J. Heat Mass Transfer. Vol. 20, pp 153–157. Pergamon Press 1977 Printed in Great Britain

BUOYANCY-INDUCED FLOW ARISING FROM A LINE THERMAL SOURCE ON AN ADIABATIC VERTICAL SURFACE

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(*Received* 9 *January* 1976)

Abstract—A study of the laminar natural convection flow arising from a steady line thermal source positioned at the leading edge of a vertical adiabatic surface is carried out. The resulting two-dimensional boundary-layer flow is analyzed and the governing equations solved numerically for a Prandtl number range of 0.01–100. The dependence of the surface temperature, the velocity level, the boundary region thickness and other physical aspects of the flow on the Prandtl number is determined. The numerical results obtained allow an evaluation of the velocity and temperature fields in the generated flow. The results are also compared with those obtained for a freely rising plane plume and for a vertical isothermal surface. Several interesting features concerning this and similar flows are brought out.

NOMENCLATURE

C_p, specific heat of the fluid at constant pressure;
f, nondimensional stream function, defined in (2);
$F(x)$, local flow rate in the boundary layer;
G, Grashof number, defined in (2a);
g, acceleration due to gravity;
I, nondimensional integral, defined in (4);
I_1, nondimensional integral, defined in (10);
k, thermal conductivity of the fluid;
N, n, constants defined in (3);
Pr, Prandtl number of the fluid, $Pr = \frac{\mu C_p}{k}$;
Q, convected thermal energy in the boundary layer;
Q_0, thermal input per unit length of the line source;
t, temperature at a point in the boundary region;
t_0, surface temperature;
t_∞, temperature of the ambient medium;
u, v, velocity components in the x, y directions;
U_c, characteristic velocity, $U_c = \nu G^2/4x$;
x, distance along the vertical surface from the line source;
y, horizontal distance from the surface.

Greek symbols

β, coefficient of thermal expansion of the fluid;
η, similarity variable, defined in (2);
δ_V, velocity boundary-layer thickness;
δ_T, thermal boundary-layer thickness;
μ, coefficient of viscosity of the fluid;
ρ, density of the fluid;
ν, kinematic viscosity of the fluid, $\nu = \mu/\rho$;
ϕ, dimensionless temperature, $\phi = \frac{t-t_\infty}{t_0-t_\infty}$;
τ, shear at the surface.

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INTRODUCTION

A PROBLEM frequently encountered in technology, particularly in electronic circuitry, is that of devices which dissipate energy at a constant rate mounted on an unheated surface. The removal of this energy is often essentially only by natural convection and it is important to determine the nature of heat transfer and of the resulting flow. These considerations relate to the arrangement of electronic components and circuit boards for effective removal of the dissipated energy. Since much of the restriction in close packing of circuitry is due to the heat-transfer considerations, it is important to determine the downstream effects of a heated body located on an unheated surface. Similar considerations are important in several manufacturing processes in which selective and local heating gives rise to a constant thermal source on an unheated surface.

The present study considers the steady laminar buoyancy-induced flow arising from a steady thermal input by a long horizontal concentrated (or line) source imbedded on a vertical insulating surface, a condition frequently approximated in practical applications. The resulting two-dimensional flow is analyzed with the boundary-layer approximations. The system of coupled differential equations is numerically solved for a Prandtl number (Pr) range of 0.01–100. Though Pr values of 0.7 and 6.7, being those of air and water, are most common, other values are also often encountered, particularly in manufacturing processes. The results are, therefore, obtained over a wide range of Pr and compared with previous results for flow adjacent to a heated vertical surface and in a freely rising plane plume. The numerical results allow a straightforward determination of the resulting temperature on the adiabatic surface, the flow velocities in the boundary layer, the extent of the boundary region and other important physical quantities.

ANALYSIS

Consider a steady thermal input at $x = 0$ on a vertical surface ($y = 0, x \geqslant 0$) in an extensive unstratified medium at a uniform temperature t_∞. The surface is taken as adiabatic and its resulting local temperature is denoted by $t_0(x)$. Since the surface is adiabatic, no loss occurs downstream of the line source of strength Q_0. Therefore, the convected energy $Q(x)$ must be the same at all $x > 0$.

$$Q(x) = \int_0^\infty C_p(t-t_\infty)\rho u \, dy = Q_0. \tag{1a}$$

Employing the notation of Gebhart [1], the generalized temperature ϕ, the similarity variably η, and the generalized stream function f are:

$$\phi = \frac{t-t_\infty}{t_0-t_\infty}, \quad \eta = \frac{y}{\delta} = \frac{yG}{4x}, \quad \psi = \nu G f \tag{2}$$

with,

$$u = \psi_y, \ v = -\psi_x, \ G = 4\cdot\left[\frac{g\beta x^3(t_0-t_\infty)}{4\nu^2}\right]^{1/4}. \tag{2a}$$

Therefore, $u = U_c f'$, where $U_c = \nu G^2/4x$ and, assuming constant properties, we have

$$Q(x) = \rho C_p(t_0-t_\infty)U_c\frac{4x}{G}\int_0^\infty \phi f' \, d\eta. \tag{1b}$$

Consider the power law variation

$$t_0 - t_\infty = Nx^n \tag{3}$$

where N and n are constants described and discussed by Gebhart [2]. From (1b) we find

$$Q(x) = (64g\beta N^5\mu^2\rho^2C_p^4)^{1/4}Ix^{(5n+3)/4}$$

where

$$I = \int_0^\infty \phi f' \, d\eta. \tag{4}$$

Therefore, $Q(x)$ is independent of x for $n = -3/5$ and the temperature at the surface decays as $x^{-3/5}$. Also

$$t_0 - t_\infty = Nx^{-3/5} = \left[\frac{Q_0^4}{64g\beta\rho^2\mu^2C_p^4I^4}\right]^{1/5} x^{-3/5}. \tag{5}$$

The differential equations, for $n = -3/5$, with the Boussinesq approximation and neglecting the pressure and viscous dissipation terms, are the same as for a plane plume, in the formulation (2).

$$f''' + (12/5)ff'' - (4/5)f'^2 + \phi = 0 \tag{6}$$

$$\phi'' + (12/5)Pr(f\phi)' = 0. \tag{7}$$

From the no-slip condition at the surface, $f(0) = f'(0)$. Also, from the definition of ϕ and for an adiabatic surface, $\phi'(0) = 1-\phi(0) = 0$. For the fifth-order system (6)–(7), one more boundary condition is needed and that arises from the condition of zero tangential velocity in the ambient medium, $f'(\infty) = 0$. Therefore, the boundary conditions are:

$$f(0) = f'(0) = \phi'(0) = 1-\phi(0) = f'(\infty) = 0. \tag{8}$$

As discussed by Gebhart, Pera and Schorr [3] for a plane plume, it can also be shown, for the present problem, that the $\phi(\infty) = 0$ condition is not independent of (8) but is automatically satisfied by them. Thus, the present system of equations differs from that for a line source plume only in the no-slip boundary condition $f'(0) = 0$, instead of $f''(0) = 0$, due to zero shear stress.

The vertical component of velocity u, the local volume rate of flow $F(x)$ and the shear at the wall, τ, are given by:

$$u = \left(\frac{2g\beta Q_0}{C_p\mu^{1/2}\rho^{1/2}I}\right)^{2/5} x^{1/5}f'(\eta) \tag{9}$$

$$F(x) = \int_0^\infty u \, dy = \nu G\int_0^\infty f' \, d\eta = \nu G I_1, \quad \text{where } I_1 = \int_0^\infty f' \, d\eta \tag{10}$$

$$\tau(x) = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\cdot\frac{\nu G^3}{16x^2}\cdot f''(0). \tag{11}$$

RESULTS AND DISCUSSION

The fifth-order system of differential equations governing the flow, (6)–(8), was solved numerically, employing a fourth-order Runge–Kutta integration scheme over the Prandtl number range 0.01–100. The profiles of temperature ϕ and velocity f' are shown in Fig. 1 for $Pr = 0.7$, and in Fig. 2 for $Pr = 6.7$. The corresponding calculated curves for a line source plume, from Gebhart *et al.* [3], and the velocity profiles for an isothermal, $n = 0$, vertical surface are also shown for comparison. Zimin and Lyakhov [4] have also considered this flow, following a formulation similar to that of Fujii [5], which employs a normalization as a boundary condition. Numerical results were obtained for $Pr = 7.0$.

The application of these results lies in the determination of the velocity level in the boundary layer, the temperature at the surface for a given thermal input, the flow rate and the difference in the flow from the corresponding plume flow. In practical applications, the determination of the temperature and flow fields downstream of the heated body is important.

The profiles in Figs. 1 and 2 are found to be quite similar in form to those for natural convection flow over a heated isothermal vertical surface. The peak velocity occurs at η values which are very close and the boundary region thickness for velocity δ_v, as well as for temperature δ_T, was found to be very close, in fact, only about 5–10% different. The thinning effect on the thermal layer as Pr increases also follows the

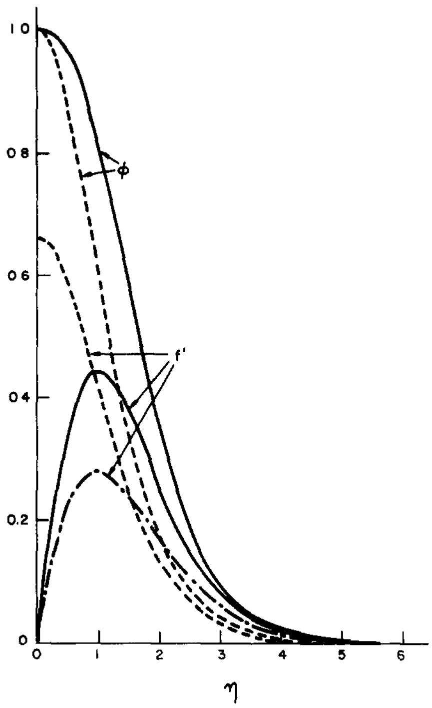

FIG. 1. Velocity, f', and temperature, ϕ, distributions at $Pr = 0.7$. —, present problem; – – –, plane plume; –·–·–, vertical isothermal surface.

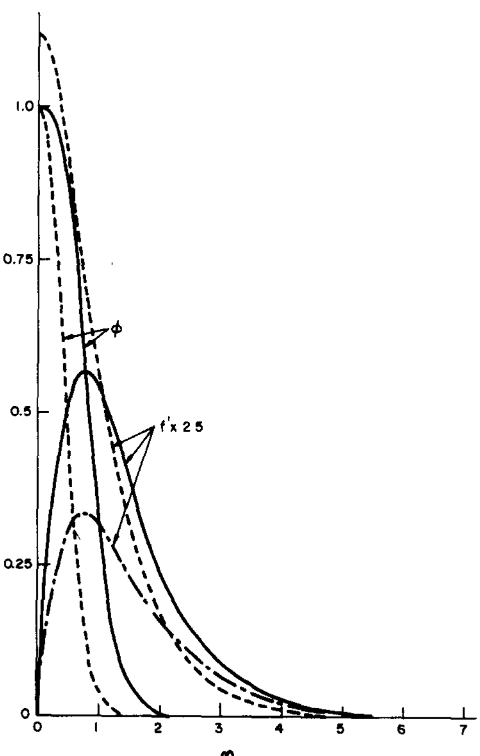

FIG. 2. Velocity, f', and temperature, ϕ, distributions at $Pr = 6.7$. —, present problem; – – –, plane plume; –·–·–, vertical isothermal surface.

same trend. However, there is a considerable difference in the peak velocity f'_{max} and in the shear stress at the wall $f''(0)$. The peak velocity in the present circumstance is substantially higher than the corresponding values for an isothermal vertical surface at the same local G, see Ostrach [6]. This difference increased from 49% at $Pr = 0.01$ to about 77% at $Pr = 100$. The shear at the wall was also found to be higher in the present case, being about 32% higher at $Pr = 0.01$ and about 50% at $Pr = 100$. This is explained physically as follows.

For a given G at a particular location x, the local temperature difference across the boundary region, $t_0 - t_\infty$, is the same for this flow as for an isothermal surface. However, since, in the present problem, the temperature at the surface decreases as $x^{-3/5}$, the upstream temperature at the surface is higher than that for the isothermal surface. This implies a higher temperature level in the flowing fluid and, hence, a greater buoyancy, which leads to higher physical velocity. The velocity being higher is seen in a higher value of f'_{max}.

A comparison of our results with those for a freely rising line source plume also show some interesting differences. It is seen from Figs. 1 and 2 that the velocity level is greatly reduced by the presence of the wall, as expected from the drag the stationary wall exerts on the flow. Another effect is the larger boundary region thickness, compared to a freely rising plume. This is, perhaps, again due to the retarding effect on the flow by the surface, causing the fluid to generate a larger normal component of velocity to spread out. The velocity level is lowered and the boundary region thickened. The overall effect on the flow rate is, therefore, expected to be small for the Pr values shown in Figs. 1 and 2.

All these considerations are shown, for the Pr range studied, in Fig. 3. As mentioned earlier, the surface temperature varies as

$$t_0 - t_\infty = Nx^{-3/5} \text{ and } N \propto \left(\frac{Q_0}{I}\right)^{4/5}$$

$$\text{where } I = \int_0^\infty \phi f' \, d\eta.$$

This integral I, which determines the velocity level and the surface temperature, was calculated over the Prandtl number range. Figure 3 indicates its value as compared to that for a two-dimensional plume, which dissipates Q_0 on either side of the centerline. The difference is small at low Pr, but becomes considerable at large Pr values. Since I is lower than for a plume, the value of N is greater than that for the plume, at given x and Q_0. Recall that $Q(x) = Q_0$. Since the velocity level is lowered by the shear at the vertical surface, the temperature level must increase, which implies a higher value of N and, hence, a higher surface temperature.

The peak physical velocity u_{max} is given in terms of f' and I, for given x and Q_0, as

$$u_{max} \propto \frac{f'}{I^{2/5}}.$$

This velocity is considerably lower than for a plume.

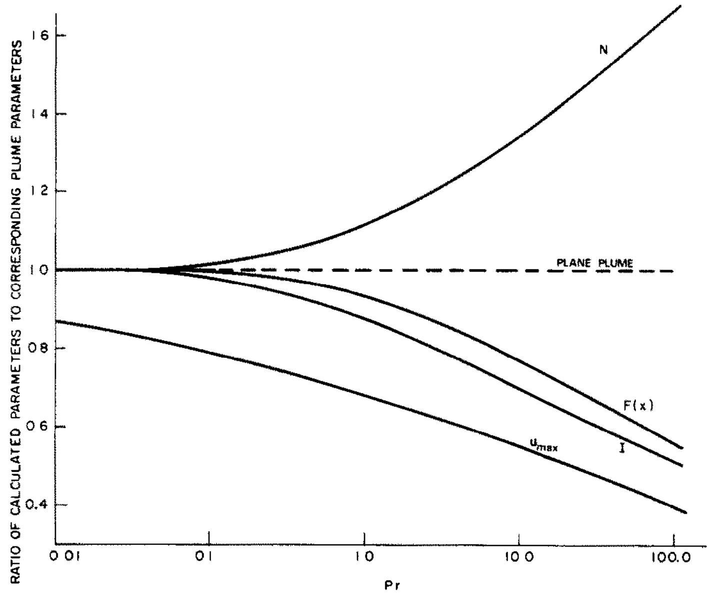

FIG. 3. Comparison of the computed values of N, $F(x)$, I and u_{max}, for the present problem, to those for a plane plume, as functions of the Prandtl number.

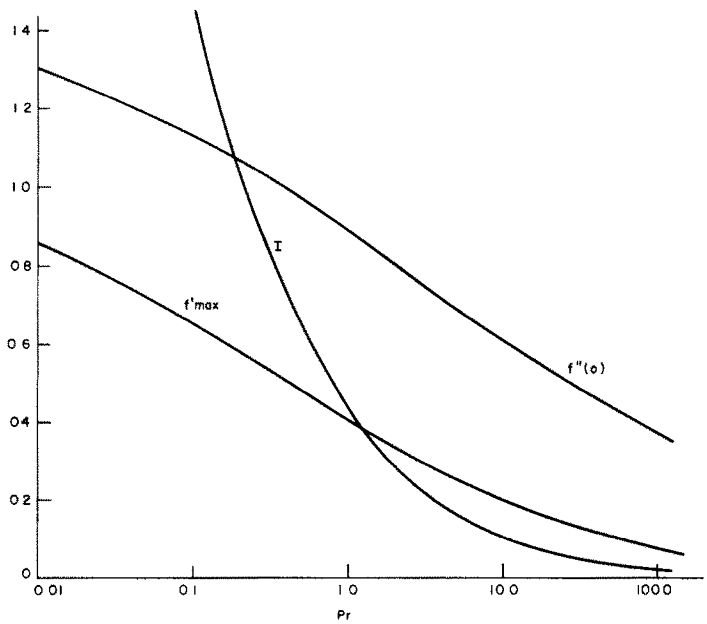

FIG. 4. Computed values of f'_{max}, I and $f''(0)$, for a line thermal source on an adiabatic surface, as functions of the Prandtl number.

The difference increases from about 13 to 60% as Pr varies from 0.01 to 100. This lowering is due to the shear at the surface. The flow rate was also calculated from the integral I_1 defined in (10). Though the difference is less than 20% up to $Pr = 6.7$, as expected from the profiles in Figs. 1 and 2, it increases to 44% at $Pr = 100$. All these differences increase with Prandtl number, as they did for a vertical isothermal surface. Therefore, the boundary condition at the surface has a greater effect as the thermal boundary region thins with increasing Pr.

As mentioned earlier, important considerations in this flow concern the velocity level, the surface temperature and the extent of the boundary region. As the flow proceeds downstream from a heated element located on an unheated vertical surface, it affects the cooling characteristics of any other elements it may encounter. An element downstream is immersed in a flowing heated fluid, whose temperature and velocity are determined by the distance between the two elements and the power input by the upstream element Q_0. If the downstream element is not dissipative it is merely heated to the corresponding surface temperature at that location, under steady state conditions, assuming it to be imbedded in the surface.

We may determine the surface temperature as

$Nx^{-3/5}$ and the velocity field in terms of f'. Figure 4 shows the necessary values, of $f''(0)$, f'_{max} and I, over the Pr range considered, to allow an evaluation of the temperature and velocity fields at a downstream element.

CONCLUSIONS

The laminar buoyancy-induced flow generated by a line thermal source imbedded in an adiabatic vertical surface is analyzed and numerical results are obtained over a Prandtl number range of 0.01–100. This flow configuration is of considerable importance in technology and the present work considers the resulting boundary-layer flow in order to determine the surface temperature, the velocity level, the boundary region thickness and other physical aspects of interest. The results obtained are compared with those for an isothermal vertical surface and with those for a freely rising plane plume. Several interesting similarities and differences are noted. The dependence of the present flow on the Prandtl number and the basic features of the flow are discussed.

The results of the present study are important in technology, for example, in the positioning of components dissipating energy on vertical circuit boards and in the positioning of the boards themselves. Heat transfer and natural convection flow considerations are crucial in this area and also in several frequently encountered manufacturing processes. The interaction of the flows arising from several elements, which constitute steady thermal sources, on a vertical insulating surface is an important problem and further work needs to be done on it in order to determine the nature of the resulting heat transfer and flow. The present study determines the flow arising from an isolated line thermal source on an adiabatic surface, and, therefore, indicates the nature of analysis for the flow arising from an interaction of several elements.

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CONVECTION NATURELLE-ECOULEMENT ASCENDANT INDUIT LE LONG D'UNE SURFACE ADIABATIQUE VERTICALE, PAR UNE SOURCE THERMIQUE RECTILIGNE

Résumé—On étudie la convection laminaire ascendante à partir d'une source thermique rectiligne placée au bord d'attaque d'une surface adiabatique verticale. L'écoulement de couche limite bidimensionnel est analysé et les équations sont résolues numériquement pour un nombre de Prandtl allant de 0,01 à 100. On dégage l'influence du nombre de Prandtl et de la température pariétale sur le niveau de vitesse, l'épaisseur de la couche et les autres aspects physiques de l'écoulement. Les résultats numériques obtenus donnent une évaluation des champs des vitesses et des températures dans l'écoulement généré. Les résultats sont comparés à ceux d'un panache vertical libre et d'une surface verticale isotherme. On dégage quelques particularités intéressantes de ces écoulements.

AUFTRIEBSSTRÖMUNG VON EINER LINIENFÖRMIGEN WÄRMEQUELLE AUF EINER ADIABATEN, VERTIKALEN OBERFLÄCHE

Zusammenfassung—Es wurde die, von einer stationären, linienförmigen Wärmequelle an der Anströmkante einer adiabaten, vertikalen Oberfläche ausgehende laminare, natürliche Konvektionsströmung untersucht. Die sich ergebende zweidimensionale Grenzschichtströmung wird analysiert und die den Vorgang beschreibenden Gleichungen werden numerisch gelöst für den Bereich der Prandtl–Zahlen von 0,01 bis 100. Die Oberflächentemperatur, das Geschwindigkeitsniveau, die Grenzschichtdicke und andere Eigenschaften der Strömung werden in Abhängigkeit von der Prandtl–Zahl bestimmt. Die numerisch ermittelten Ergebnisse erlauben eine Bestimmung des Geschwindigkeits- und Temperaturfeldes in der so erzeugten Strömung. Die Ergebnisse werden außerdem mit jenen für eine frei aufsteigende, ebene Auftriebsströmung bzw. für eine Auftriebsströmung an einer vertikalen, isothermen Fläche verglichen. Mehrere interessante Merkmale dieser und ähnlicher Strömungen werden aufgedeckt.

ВОЗНИКНОВЕНИЕ ТЕЧЕНИЯ ЗА СЧЕТ ПОДЪЕМНЫХ СИЛ ОТ ЛИНЕЙНОГО ИСТОЧНИКА ТЕПЛА НА АДИАБАТИЧЕСКОЙ ВЕРТИКАЛЬНОЙ ПОВЕРХНОСТИ

Аннотация — Исследуется возникновение ламинарного течения при естественной конвекции от стационарного линейного источника тепла, расположенного на передней кромке вертикальной адиабатической поверхности. Анализируются и численно решены основные уравнения двумерного пограничного слоя для чисел Прандтля от 0,01 до 100. Найдена зависимость температуры поверхности, скоростей, толщины пограничного слоя и других физических параметров потока от числа Прандтля. Полученные численные результаты позволяют определить профили скорости и температуры в потоке. Результаты сравниваются с ранее полученными данными для свободной плоской струйки и для вертикальной изотермической поверхности. Рассматриваются некоторые особенности данного и аналогичного течений.